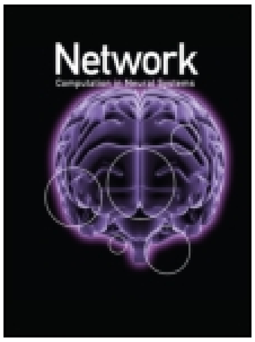

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Network: Computation in Neural Systems

ISSN: 0954-898X (Print) 1361-6536 (Online) Journal homepage: informahealthcare.com/journals/inet20

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To cite this article: M Bauer & W Martienssen (1991) Coupled circle maps as a tool to model synchronisation in neural networks, Network: Computation in Neural Systems, 2:4, 345-351, DOI: 10.1088/0954-898X_2_4_001

To link to this article: https://doi.org/10.1088/0954-898X_2_4_001

Network 2 (1991) 345–351. Printed in the UK

LETTER TO THE EDITOR

Coupled circle maps as a tool to model synchronisation in neural networks

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Abstract. Temporal correlation and decorrelation of the spiking of groups of neurons have been suggested to be of importance for the segmentation of different features to objects (binding problem). We show that coupled circle maps exhibiting chaotic oscillations are a useful tool to simulate the behaviour of such systems. In a model where one map represents the phase dynamics of one neuron or a group of neurons we observe that, depending on the coupling strength, the different maps show correlated or uncorrelated behaviour, while the autocorrelation function remains flat, as expected for a chaotic signal. This synchronized behaviour can be organized by a simple Hebb-type learning rule.

It has been claimed [1] that besides the spiking frequency of the neural activity, the 'phase' of the neural oscillation plays an important role in the processing of data in the brain. The correlation of the spikes of different active neurons is suggested to code whether or not different signals belong to the same object (binding problem). Recent experiments [2–4] support these considerations by finding correlated firing of neurons that correspond to different receptive fields when these fields are stimulated by the same object. These features are normally simulated using neural network models consisting of coupled oscillators, mostly relaxation oscillators [5–13].

The dynamics of nonlinear oscillators has been subject of extensive studies [14, 15]. Circle maps [16] have proven to be very useful for the description and understanding of such systems, namely of uncoupled relaxation oscillators [17, 18]. It has also been shown that it is possible to reduce the dynamics of a leaky integrator model [19] for one neuron to a circle map. Beyond the studies on single maps the behaviour of *coupled* maps with various topologies has been treated in terms of nonlinear dynamics [20–22].

In this letter we point out that coupled circle maps can be used to simulate the correlation of phases, even when the signal of the single neuron is chaotic. The coupling method used here is based on the method described in [23] for the determination of Lyapunov exponents by studying correlation between the chaotic motions of two equivalent coupled systems. The organization of neurons in different groups, showing synchronous behaviour within one group, can be demonstrated using a simple Hebb-like learning rule. The aim of this paper is to illustrate a way for the study of dynamical properties and synchronization in neural systems; we do not attempt to make a neurophysiological model of, for example, the visual system nor to present a technical system for object recognition.

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0954-898X/91/040345+07$03.50

The N neurons of the considered model are described by two variables the phase θ_i and the activity s_i. The dynamics of the phases θ_i is modelled using standard sine circle maps†, with an additional noise term η that represents equally distributed random numbers in the interval $[0, \eta']$. The sine circle map $\varphi(x)$ is defined by:

$$\varphi(x) = x + \Omega + \frac{k}{2\pi}\sin(2\pi x) + \eta \qquad (\text{mod } 1). \tag{1}$$

Using this definition of φ the new phase $\theta_i(t+1)$ of the ith neuron is calculated:

$$\theta_i(t+1) = \frac{1}{1+\kappa}[\varphi(\theta_i(t)) + \kappa\varphi(\vartheta_i(t))]. \tag{2}$$

The circle map φ is applied on the old phase $\theta_i(t)$ and—weighted with a coupling strength κ—on an input value ϑ_i. This input is given according to the phases of the other neurons weighted with a coupling matrix J ($J_{ii} = 0, \forall i$):

$$\vartheta_i(t) = \frac{\sum_j J_{ij}\theta_j}{\sum_j J_{ij}}. \tag{3}$$

In our simulations we use the parameters‡ for the circle map $k = 5$, $\Omega = 0.618$, that assure chaotic oscillations of the uncoupled map with a Lyapunov exponent of $\lambda = 0.89$.

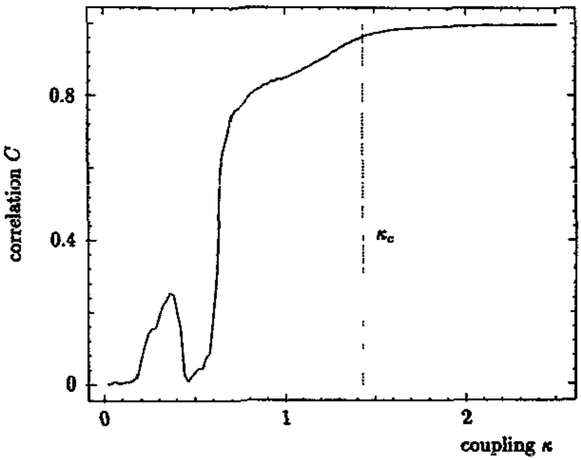

Figure 1. The cross-correlation $C(0)$ of the phases in a network of $N = 100$ coupled neurons ($k = 5$) is plotted versus the coupling strength κ. We observe that at the critical coupling strength $\kappa_c = 1.43$ the solution with $C(0) = 1$ loses its stability.

In figure 1 we plot the averaged (over 20 different initial conditions) cross correlation C between two coupled neurons in a network of size $N = 100$ versus the

† Similar results can be achieved with other circle maps, e.g. the maps described in [19].

‡ The choice of these parameters is not critical.

coupling strength κ ($J_{ij} = 1, \forall i \neq j$). We observe that the strongly correlated state loses its stability below a critical coupling strength around $\kappa_c = 1.5$.

This change of the correlation does not correspond to a transition from a periodic or stationary behaviour at high κ-values to chaotic oscillations below κ_c. With the exception of the little peak around $\kappa = 0.3$ the oscillation is always chaotic.

In order to study the stability of the strongly correlated state ($\theta_i(t) = \theta_j(t), \forall i, j$), we define the average phase $\vartheta(t) = 1/N \sum \theta_j$. Neglecting (in a large network) the contribution of θ_i to ϑ we can write:

$$\theta_i(t+1) = \frac{1}{1+\kappa}[\varphi(\theta_i(t)) + \kappa\varphi(\vartheta(t))] \tag{4}$$

$$\vartheta(t+1) = \varphi(\vartheta(t)) . \tag{5}$$

Considering the dynamics of the difference $\Psi_i(t) = \theta_i(t) - \vartheta(t)$ between the phase of one neuron and the average phase, we find:

$$\Psi_i(t+1) = \frac{1}{1+\kappa}[\varphi(\theta_i(t) + \kappa\varphi(\vartheta(t))] - \varphi(\vartheta(t)) . \tag{6}$$

Linear stability analysis of the phase difference gives:

$$\Psi_i(t+1) = \frac{1}{1+\kappa} \frac{\mathrm{d}\varphi(\vartheta(t))}{\mathrm{d}\vartheta} \Psi_i(t) . \tag{7}$$

Using the definition of the Lyapunov exponent for the uncoupled map

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} \frac{\mathrm{d}\varphi(x_i)}{\mathrm{d}x} \right| \tag{8}$$

the correlated solution loses its stability at the critical coupling strength† κ_c:

$$\kappa_c = \mathrm{e}^{\lambda} - 1 . \tag{9}$$

This allows us to calculate the parameter κ for our simulation directly from the properties of the one dimensional circle map. Applying (9) on the simulation parameters used in figure 1 we find a value of $\kappa_c = 1.43$ which is in good agreement with the numerical result.

In figure 2 the time dependence of the phase variables θ_i in a network of $N = 10$ neurons for a coupling strength $\kappa = 1.5$ and a small noise amplitude $\eta = 10^{-6}$, is plotted for different coupling situations. In the first section (A) of figure 2 the ten neurons are uncoupled (i.e. $J_{ij} = 0, \forall i, j$), which results in totally uncorrelated chaotic behaviour. In the second section (B) the coupling is switched on ($J_{ij} = 1, \forall i \neq j$), after a few iterations the phases of the neurons correlate showing the same chaotic time series. In the following section (C) of the figure we divide the neurons in two groups, i.e. the neurons within both groups are totally coupled but there is no connection between the groups. Coupled neurons again show the same chaotic time series, but due to the noise in the system the two groups decorrelate

† This solution holds only for large networks, in smaller networks the critical coupling lies between this value and the solution for *two* coupled maps: $\lambda = \ln((1+\kappa_{c2})/(1-\kappa_{c2}))$.

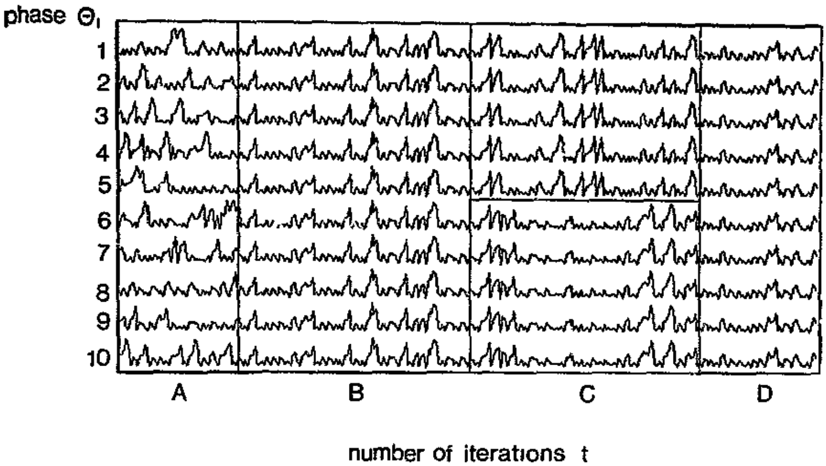

Figure 2. The time dependence of the phases θ_i in a network of $N = 10$ neurons with the parameters $k = 5$, $\kappa = 1.5$ is plotted. The vertical lines mark the times when the coupling is changed. (A) All neurons are uncoupled $J_{ij} = 0$. (B) All neurons are coupled $J_{ij} = 1$ for $i \neq j$. (C) The neurons are divided into two groups (the separation of the groups is marked by the horizontal line) Only the neurons within each group are coupled. (D) The neurons are coupled as in (B).

after a few iterations even though the noise amplitude is very small. A small† noise amplitude is necessary if a zero coupling between the groups is chosen, in order to allow identical starting phases to show a different time evolution. Alternatively or additionally the coupling between the groups can be made slightly negative (e.g. $J_{ij} = -0.1$) which allows the groups to decorrelate without noise. In the last part (D) of this figure the coupling between the two groups is switched on again and we observe that the neurons correlate again very quickly.

In order to show the correlation properties of the system in a more quantitative way and to demonstrate that the presented features are not an effect of transients or only of very small networks, we calculate the cross-correlation function‡ of a network with $N = 1000$ neurons devided into two groups averaged over 10000 iterations. The result of this simulation is depicted in figure 3. We find that the correlation between two neurons within the same group is flat at $C(\tau) = 0$ except the peak around $\tau = 0$ where the correlation becomes one. This result is identical with the autocorrelation function of one of the chaotic signals. If we consider the phases of neurons in different groups, we observe that the correlation function remains flat around $C(\tau) = 0$ even at the delay $\tau = 0$.

† In fact the noise amplitude can be chosen arbitrarily small because the difference between the two groups will grow exponentially. For numerical simulations it must be chosen large enough that the contribution of the noise is not rounded during the iteration.

‡ The temporal correlation between two time series X_t, Y_t defined by

$$C(\tau) = \frac{\sum x_i y_{i+\tau}}{\sqrt{\sum x_i^2 \sum y_{i+\tau}^2}}$$

with $x_i = X_i - \langle X \rangle$ and $y_i = Y_i - \langle Y \rangle$ where $\langle \cdot \rangle$ denotes the time average.

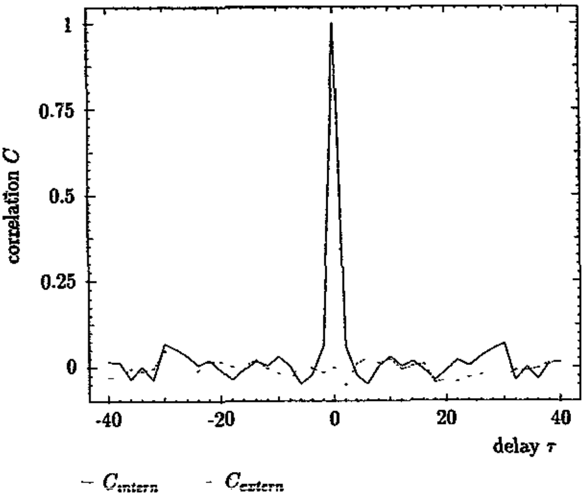

Figure 3. The cross-correlation function $C(\tau)$ is plotted for two neurons that are in the same group (solid line) and for two neurons that are in different groups (broken line).

In the last part of this letter we give an example for a simple learning algorithm that demonstrates how the neural network can organize itself into groups. In order to do this, we present the different groups (objects) randomly to the network and apply a Hebb-like learning rule for the coupling strengths J_{ij}

$$J_{ij}(t+1) = \Phi[J_{ij}(t) - \lambda + \gamma s_i s_j] \tag{10}$$

where γ determines the learning speed and λ is a 'forgetting'–term. The function $\Phi(x)$ confines the value of the coupling strength in the range $[\theta_{\min}, \theta_{\max}]$:

$$\Phi(x) = \begin{cases} x & \text{for } \theta_{\min} \leqslant x \leqslant \theta_{\max} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

For our simulations we choose $\lambda = 0.001, \gamma = 0.01, \theta_{\min} = 0$ and $\theta_{\max} = 1$. In every timestep during the learning phase each group is independently chosen to be active with a probability p_a. That means with a probability p_a we set the activity s_i of all neurons belonging to one group to $s_i = 1$.

This learning process is shown in figure 4. We want the network ($N = 12$) to organize in three independent groups of synchronous phase. According to the learning algorithm described above we start at $t = 0$ to present the different objects (groups) in a random fashion with $p_a = 0.3$ to the network (the presented patterns are shown at the bottom of the figure). After this learning, the neurons have organized into three groups even though these groups have not always been presented separately during the learning, i.e. the network distinguishes the neurons that are always activated simultaneously because they belong to the same object and those that are only sometimes activated simultaneously because they belong to different objects that are presented independently at the same time. In that way the network has learned three different 'objects' that can be distinguished even when they are presented at the same

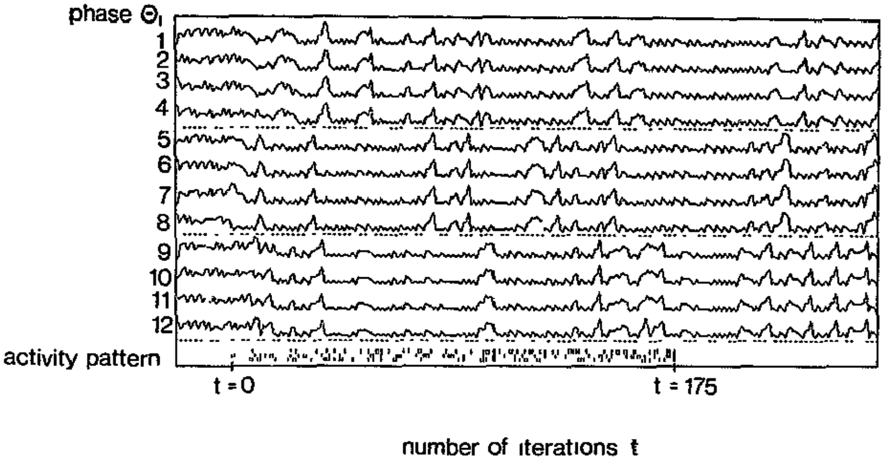

Figure 4. Learning in a network of $N = 12$ neurons; the phases θ_i of the twelve neurons are shown in the upper part of the figure. In the last row the presented activity pattern is shown. The learning starts at $t = 0$ and ends at $t = 100$. After this the couplings have arranged according the 'learning rule' (10), leading to a configuration that consists of three regions that show independent chaotic oscillations (for clarity the three regions are separated by the dotted lines)

time. The neurons of one group continue to oscillate in phase for a time depending on the 'forgetting' constant λ.

In conclusion we have demonstrated that coupled circle maps are a useful tool to simulate and describe the correlation behaviour of neural networks. Besides the fact that one-dimensional maps can be simulated more efficiently than differential equations, one can profit from knowledge about the dynamical behaviour of circle maps. The possibility to choose chaotic states of the maps allows us to distinguish a large number of objects (i.e. build a number of different groups if the network is sufficiently large) that show no correlation between the neurons belonging to different objects but perfect correlation within the same group. Practically the number of different groups is limited by the resolution of the phase variable and the length of the considered time interval.

This work is supported by the Deutsche Forschungsgemeinschaft via the Sonderforschungsbereich 185 'Nichtlineare Dynamik'.

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